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LEFT QUASI NOETHERIAN RINGS AND MODULES

A thesis submitted in Partial Fulfillment of the Requirements of the Award of the M. Sc. Degree in Mathematics

By

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CONTENTS

INTRODCTION AND SUMMARY

One of the important problems of Modules and Rings theory is the determination of the structure of Modules (Rings) satisfies the ascending chain condition on submodules (left ideals). Such Modules (Rings) are usually called *left Noetherian Modules* (*Rings*) after E-Noether (١٨٨٢-١٩٣٥) who first realized their importance. The successful study of such classes of Modules and Rings pointed out that a theory of more general Modules (Rings) is possible. In this work we study those Modules (Rings) which satisfies certain chain condition of ascending type , leading to a new class of left Quasi-Noetherian Modules (Rings) which possesses several of the main properties of left Noetherian Modules (Rings). The condition required is the following .

Let R be a ring , and let M be a left R-module , we say that M is a left Quasi-Noetherian if for every ascending chain $N_١ \subseteq N_٢ \subseteq \dots \subseteq N_n \subseteq \dots$ of R-submodules of M , there exists $m \in Z^+$ such that $R^m(\bigcup_n N_n) \subseteq N_m$.

If $_R R$ is a left Quasi-Noetherian we say that R is a left Quasi-Noetherian ring . This is a generalization of left Noetherian and nilpotent rings.

Chapter one is devoted to give a number of basic definitions, some well-known results which are needed through this work.

Chapter two of this thesis consists of three sections. *In the first one* we give definitions , examples of a left Quasi-Noetherian Rings and Modules. Then we consider the problem of finding some conditions which are equivalent to the definitions . Inparticular we prove (Theorem ٢,١,٤) which state that : if M is a left R-module , then the following conditions are equivalent :

(a) M is a left Quasi-Noetherian.

(b) In each non-empty collection Γ of R-submodules of M , there is an element N and an $m \in Z^+$ such that $R^m K \subseteq N$ for every $K \in \Gamma$ and $K \supseteq N$.

(c) In each non-empty collection Γ of R-submodules of M such that $K \in \Gamma \Rightarrow K :_M R = \{m \in M \mid Rm \subseteq K\} \in \Gamma$, there exists a maximal element.

(d) For every ascending chain of R-submodules $N_١ \subseteq N_٢ \subseteq ... \subseteq N_n \subseteq ...$ there exists $m \in Z^+$ such that the ascending chain $N_١ :_M R^m \subseteq N_٢ :_M R^m \subseteq ... \subseteq N_n :_M R^m \subseteq ...$ of R-submodules terminates.

Next we study the relation between left Noetherian and left Quasi-Noetherian Modules, inparticular we show that : If RM is a left Noetherian , then M is a left Quasi-Noetherian Module (Theorem

٢,١,٦). Then we show that the class of left Quasi-Noetherian Modules is S-closed, Q-closed (Theorem ٢,١,٨) and E-closed (Theorem ٢,١,١٢) while the class of left Quasi-Noetherian Rings is neither S-closed nor E-closed, but it is I-closed (Theorem ٢,١,١٥) and Q-closed (Theorem ٢,١,٨) . We show also that a finite direct sum of left Quasi-Noetherian rings is a left Quasi-Noetherian ring (Theorem ٢,١,١٧) . Finally we prove (Theorem ٢,١,١٨) which state that : If R is a left Quasi-Noetherian ring , then R satisfies the ascending chain condition on semi-prime ideals.

In section two we study the ideal structure and we give some classification . First we prove that If R is a left Quasi-Noetherian ring , then every nil left ideal of R is nilpotent (Theorem ٢,٢,١). Then we classify the class of left Quasi-Noetherian rings, inparticular we prove the following : The ring R is a left Quasi-Noetherian if and only if the nil radical $N=N(R)$ is nilpotent and each of the R-modules R/N, $N/N^{٢}$, $N^{٢}/N^{٣}$, ... is a left Quasi-Noetherian (Theorem ٢,٢,٤) . Next we prove that : If R is a commutative Quasi-Noetherian ring , then every irreducible ideal of R is Primary (Theorem ٢,٢,٥) . Then we prove that every ideal in a commutative Quasi-Noetherian ring is contained in a finite intersection of primary ideals (Theorem ٢,٢,٦) . Next we study the existence of prime ideals in the left Quasi-Noetherian rings, Inparticular,

we show that If R is a non-nilpotent left Quasi-Noetherian ring then there exists a prime ideal in R which is a right annihilator of some non-zero ideal in R (Theorem ٢,٢,٨) . Finally we prove (Theorem ٢,٢,٩) which assert that every ideal in a left Quasi-Noetherian ring contains a product of prime ideals .

In section three we study the relationship between left Quasi-Noetherian rings and some certain special types of rings. First we give its relationship with left Goldie rings , inparticular we show that if R is a left Quasi-Noetherian ring and $r(R) =$ ٠ , then R is a left Goldie ring (Theorem ٢,٣,٢) . Then we show that if R is a left Quasi-Noetherian ring with at least one regular element , then its ring of left quotient $Q(R)$ is a left Quasi-Noetherian ring (Theorem ٢,٣,٥). Next we prove (Theorem ٢,٣,٧ and Theorem ٢,٣,١١) if R is a left Quasi-Noetherian ring without zero divisors then the polynomial ring $R[x]$ and the ring of formal power series $R[[x]]$ are left Quasi-Noetherian .

Finally we remark that two papers [١, ٢] based on this work have been submitted for publication .

CHAPTER ONE

Basic concepts

In this chapter , we collect some well-known results , which are needed.

Definition ١,١,١

Let R be a ring , A, B are ideals (left or right) ideals of R, then the sum $A+B=\{a+b \mid a\in A, b\in B\}$ is an ideal (left , right) in R. And the product $AB=\{ \sum_{\text{finite}} a_i b_i \mid a_i \in A , b_i \in B \}$ is an ideal (left , right) in R .

It is clear that if A ,B are ideals, then $AB \subseteq A \cap B$ however if A, B are left ideals then $AB \subseteq B$.

Definition ١,١,٢

(a) A is said to be a *nilpotent* (*right*, *left or two sided*) *ideal* of R, if there exists an $n \in \mathbf{Z}^+$ such that $A^n =$ ٠ .

(b) A is said to be a *nil* (*right*, *left or two sided*) *ideal* of R if every element of A is nilpotent.

(c) A ring R is said to be a *semi-prime ring* if it has no non-zero nilpotent ideal.

Definition ١,١,٣

(a) An ideal P in a ring R is said to be a *prime ideal* if and only if A and B are ideals in R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

If R is a commutative ring then the ideal P is a prime if and only if for all a,b in R, $ab \in P$ implies that $a \in P$ or $b \in P$.

A ring R is said to be a *prime ring* if and only if the zero ideal $\langle ٠ \rangle$ is a prime ideal in R.

(b) An ideal Q in a ring R is said to be a *semi-prime ideal* if and only if $A^{٢} \subseteq Q$, $A \lhd R$, then $A \subseteq Q$.

Example ١,١,٤

(a) The prime ideals of the ring $\mathbf{Z}$ are precisely the ideals $\langle p \rangle$, where p is a prime number together with the two trivial ideals $\langle ٠ \rangle$ and $\mathbf{Z}$.

(b) If $R = (\mathbf{Z}[x], +, .)$ then $p = \langle x \rangle$ is a prime ideal of R.

Lemma ١,١,٥

(a) If Q is a semi-prime ideal in R and $A^n \subseteq Q$, then $A \subseteq Q$.

(b) Q is a semi-prime ideal in R if and only if R/Q is a semi-prime ring.

(c) P is a prime ideal in R if and only if R/P is a prime ring.

It follows from (c) that if R is a commutative ring with identity, then P is a prime ideal of R if and only if R/P is an integral domain.

Definition ١,١,٦

(a) The *prime radical of an ideal* I, denoted by $rad(I)$, in a ring R is the intersection of all prime ideals in R which contains I.

If $I = \langle ٠ \rangle$, then we call $rad\langle ٠ \rangle$, the prime radical of R and is usually denoted by $rad(R)$.

If R is commutative, then $rad(I) = \{ a \in R \mid a^n \in I$ for some $n \in Z^+ \} = \sqrt{I}$.

(b) The sum of all nil (right , left or two sided) ideals of a ring R is called the *nil radical of* R and denoted by $N(R)$ or $W(R)$.

Definition ١,١,٧

Let I be an ideal of the ring R. Then I is said to be *irreducible* if it is not a finite intersection of ideals of R properly containing I; otherwise, I is *reducible*.

Definition ١,١,٨

Let R be a commutative ring . An ideal I of the ring R is called *primary* if $ab \in I$ and $a \notin I$ implies that $b^n \in I$ for some positive integer n.

Lemma ١,١,٩

Let R be a commutative ring . Then

(a) I is a primary ideal of the ring R if and only if $ab \in I$ and $a \notin I \Rightarrow b \in \sqrt{I}$.

(b) I is a primary ideal in R if and only if every zero divisor of R/I is nilpotent.

Example ١,١,١٠

(a) If R is a commutative ring , then any prime ideal is primary.

(b) For any prime integer p and any positive integer n, $\langle p^n \rangle$ is a primary ideal in $\mathbf{Z}$.

Definition ١,١,١١

A prime ideal is *minimal prime ideal* of R if it does not properly contain any other prime ideal of R.

Example ١,١,١٢

(a) $R = (\mathbf{Z}_٤, \oplus, \otimes)$, then $P = \langle ٢ \rangle$ is a minimal prime ideal in R.

(b) Let $R = (\mathbf{Z}, +, .)$, then $\langle ٢ \rangle$ is not a minimal prime ideal of R , since $\langle ٠ \rangle \subsetneq \langle ٢ \rangle$ and $\langle ٠ \rangle$ is a prime ideal in R.

(c) $R = (\{ \begin{pmatrix} a & b \\ ٠ & c \end{pmatrix} \mid a,b,c \in \mathbf{Z} \} ,+, .)$, then $P = (\{ \begin{pmatrix} ٠ & a \\ ٠ & b \end{pmatrix} \mid a,b \in \mathbf{Z} \} ,+, .)$ is a minimal prime ideal in R.

Definition ١,١,١٣

Let S be a non-empty subset of a ring R , Then $r(S) = \{x \in R \mid sx = ٠ \text{ for all } s \in S\}$ is called the *right annihilator of S in R*.

If S consists of a single element a, $S = \{a\}$, we write $r(S) = r(a)$. It is clear that $r(S) \lhd_r R$.

We say that the right ideal I of R is a right annihilator if $I = r(S)$,

for some subset S of R.

The *left annihilator* $\ell(S)$ can be defined similarly.

Note that:

(a) If R is a ring with identity , then $r(R) = \ell(R) = ٠$.

(b) If R is a commutative ring , then $r(S) = \ell(S)$, and is usually denoted by $ann(S)$.

(c) If R is a semi-prime ring , and $I \triangleleft R$ .Then:

(i) $\ell(I) = r(I)$; (ii) If I is a right annihilator ideal in R , then $I = r(\ell(I)) = \ell(r(I))$ so that I is also a left annihilator ideal in R and we call I is an annihilator ideal.

Lemma ١,١,١٤

(a) $S \subseteq \ell(r(S))$, $S \subseteq r(\ell(S))$.

(b) If $S \subseteq T$, then $r(S) \supseteq r(T)$ and $\ell(S) \supseteq \ell(T)$.

(c) $I = r(S) \Leftrightarrow I = r(\ell(I))$ and $I = \ell(S) \Leftrightarrow I = \ell(r(I))$.

(d) If S is a right (left) ideal of R , then $r(S)$ ($\ell(S)$) is an ideal of R.

(e) If $S \triangleleft R$ then $r(S)$ and $\ell(S)$ are both ideal of R.

(f) If S and T are any subset of R , then :

(i) $r(S) + r(T) \subseteq r(S \cap T)$; and (ii) $r(S \cup T) = r(S) \cap r(T)$.

Definition ١,١,١٥

Let R be a ring. A (right ,left) ideal I in R is said to be a *complement* (*right* , *left*) *ideal* in R if there exists a non-zero (right , left) ideal J in

R such that $I \cap J = ٠$ and if K (right , left) ideal in R , $K \supset I$ then $K \cap J \neq ٠$.

i.e. I is maximal with the property that $I \cap J = ٠$.

Example ١,١,١٦

(a) $R = (\mathbf{Z},+,.)$, then $ann\, I = \langle ٠ \rangle$ for all $٠ \neq I \lhd R$.

(b) $R = (\mathbf{Z}_{٨},\oplus,\otimes)$, $I = \langle ٤ \rangle$ then $ann\, I = \langle ٢ \rangle$.

(c) $R = (\mathbf{Z}_{٦},\oplus,\otimes)$, then $I = \langle ٣ \rangle$ is a complement ideal of $J = \langle ٢ \rangle$.

(d) $R = (\mathbf{Z}_{١٢},\oplus,\otimes)$, then $I = \langle ٤ \rangle$ is a complement ideal of $J = \langle ٦ \rangle$

but J is not a complement ideal of I since $\langle ٦ \rangle \subsetneq \langle ٣ \rangle$ and $\langle ٣ \rangle \cap \langle ٤ \rangle = ٠$.

Definition ١,١,١٧

(a) If M is an R-module, then $A(M) = \{x \in R \mid xM = ٠\}$ is called the annihilator of M in R , and it is clear that $A(M) \lhd R$.

(b) An R-module M is said to be simple if its only submodules are ٠ and M.

If M is simple R-module and $RM = \{ rm \mid m \in M , r \in R \} \neq ٠$ then M is called an irreducible R-module.

(c) The *Jacobson radical of a ring R*, denoted by $J(R)$, is defined as $J(R) = \cap A(M)$ where M is an irreducible R-module.

If R is a ring with identity , then $J(R)$ is the intersection of all maximal right (left) ideals of R .

It is well known that $rad(R) \subseteq N(R) \subseteq J(R)$.

Example ١,١,١٨

(a) $R = (\mathbf{Z},+,.)$, then $J(R) = \bigcap \langle p \mid p \text{ prime number}\rangle = \langle ٠\rangle$.

(b) $R = (\mathbf{Z}_{٨},\oplus,\otimes)$ so $J(R) = \langle ٢\rangle$.

(c) $R = (\{\begin{pmatrix} a & b \\ ٠ & c \end{pmatrix} \mid a,b,c \in \mathbf{R}\} ,+,.)$, then

$$J(R) = (\{\begin{pmatrix} a & b \\ ٠ & ٠ \end{pmatrix} \mid a,b \in \mathbf{R}\} ,+,.) \cap (\{\begin{pmatrix} ٠ & b \\ ٠ & c \end{pmatrix} \mid b,c \in \mathbf{R}\} ,+,.)$$

$$= (\{\begin{pmatrix} ٠ & b \\ ٠ & ٠ \end{pmatrix} \mid b \in \mathbf{R}\} ,+,.) .$$

Definition ١,١,١٩

Let I and J be two ideals of the ring R. Then the *right* (*left*) *quotient of I by J*, denoted by $I :_r J$ ($I :_\ell J$), $I :_r J = \{a \in R \mid aJ \subseteq I\}$, $I :_\ell J = \{a \in R \mid Ja \subseteq I\}$.

If R is a commutative ring , we simply write $I : J$.

Note that:

(a) $I :_r J$ and $I :_\ell J$ actually form an ideal of R , whenever I and J are ideals.

(b) If I is a left ideal of a ring R then $I :_\ell R = \{ b \in R \mid Rb \subseteq I\}$ is a left ideal in R such that $I \subseteq I :_\ell R$ and $I :_r R = \{ b \in R \mid bR \subseteq I\}$ is an ideal contains every ideal of R contained in I. In other word ($I :_r R$)

is the (unique) largest ideal of R contained in I.

(c) If M is a left R-module , N is a submodule of M, then $N :_M R = \{x \in M \mid Rx \subseteq N\}$ is a submodule of M , which is called the *quotient of* N *by* R .

Example ١,١,٢٠

(a) $R = (\mathbf{Z}, +, .)$, $I = \langle ٢ \rangle, J = \langle ٦ \rangle$ then $I : J = R$, $J : I = \langle ٣ \rangle$.

(b) $R = (\mathbf{Z}_٦, \oplus, \otimes), I = \langle ٢ \rangle$, $J = \langle ٠ \rangle$ then $I : J = R$, $J : I = \langle ٣ \rangle = ann I$.

Lemma ١,١,٢١

Let I_i , J_i be ideals of R then :

(a) $(\bigcap_i I_i) :_\ell J = \bigcap_i (I_i :_\ell J)$;

(b) $I :_\ell \sum_i J_i = \bigcap_i (I :_\ell J_i)$;

(c) $I :_\ell J^{n+١} = (I :_\ell J^n) :_\ell J = (I :_\ell J) :_\ell J^n$ for any $n \in \mathbf{Z}^+$;

(d) $٠ :_r I = \ell(I)$ and $٠ :_\ell I = r(I)$;

(e) If $I \subseteq J$, then $I :_\ell K \subseteq J :_\ell K$ and $K :_\ell I \supseteq K :_\ell J$.

(f) $I :_r J$ $(I :_\ell J)$ is the largest ideal of R with the property that $(I :_r J) J \subseteq I$ $(J (I :_\ell J) \subseteq I)$.

Definition ١,١,٢٢

Let R be any ring then :

(a) An element x of R is said to be *regular* if its neither a left nor right zero divisor in R i.e. $r(x) = \ell(x) =$ ٠ .

(b)The *ring of left quotient of* R , if it exists , is a ring Q which satisfies the following conditions:

(١)R is a subring of Q .

(٢)Each regular element of R is a unit of Q .

(٣) Each element q of Q is of the form $c^{-١}a$ for some elements a and c of R with c regular (i.e. $cq \in R$ for some regular $c \in R$) .

The *ring of right quotient of* R is definite similarly (with $ac^{-١}$ instead of $c^{-١}a$). If both the ring of left quotient and right quotient exist then , we say that R has a *ring of quotient* , and we denoted by $Q(R)$.

Example ١,١,٢٣

(a) If $R = (\mathbf{Z}, +, .)$ then $Q(R) = (\mathbf{Q}, +, .)$.

(b) If $R = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ ٠ & \mathbf{Z} \end{pmatrix} = \{ \begin{pmatrix} a & b \\ ٠ & c \end{pmatrix} \mid a,b,c \in \mathbf{Z} \}$ then $Q(R) = \begin{pmatrix} \mathbf{Q} & \mathbf{Q} \\ ٠ & \mathbf{Q} \end{pmatrix}$.

(c) If $R = \mathbf{Z}(\sqrt{٢}) = \{ a + b\sqrt{٢} \mid a,b \in \mathbf{Z} \}$, then $Q(R) = \mathbf{Q}(\sqrt{٢}) = \{a + b\sqrt{٢} \mid a,b \in \mathbf{Q}\}$.

(d) If $R = (F[x], +, .)$, then $Q(R) = F\langle x \rangle = \{ \sum_{i=-\infty}^{\infty} a_i x^i \mid a_i \in F \}$.

Definition ١,١,٢٤

A left R-module M is said to be *a left Noetherian* if M satisfies the

ascending chain condition (a.c.c) on submodules of M i.e. for every ascending chain of submodules of M, $N_{١} \subseteq N_{٢} \subseteq \ldots$, there exists a positive integer n such that $N_m = N_n$ for all $m \geq n$.

The ring R is said to be a left Noetherian if ${}_R R$ is a left Noetherian module.

Example ١,١,٢٥

(a) Every field, and every finite ring is Noetherian.

(b) Every finite abelian group $(G , +)$ is a Noetherian $\mathbf{Z}$-module.

(c) Every principle ideal ring is Noetherian . Hence $(\mathbf{Z}, +, .)$, $(F[x], +, .)$,$(\mathbf{Z}(i), +, .) = \{ a + ib \mid a,b \in \mathbf{Z} \}$ are Noetherian rings.

(d) $M = F[x]$ regarded as F-module is not Noetherian module because $\langle ١ , x \rangle \subset \langle ١ ,x , x^{٢} \rangle \subset \langle ١ ,x ,x^{٢},x^{٣} \rangle \subset \ldots$ is an infinite ascending chain of submodules of M .

(e) $(\mathbf{Z}[x_{١}, x_{٢},...,x_n], +, .)$ is a Noetherian ring by Hilbert's Bases Theorem.

(f) The infinite polynomial ring $R = F[x_{١} ,x_{٢} , \ldots]$ over a field F. is an integral domain which is not Noetherian ring since $\langle x_{١} \rangle \subset \langle x_{١},x_{٢} \rangle \subset \langle x_{١},x_{٢} ,x_{٣} \rangle \subset \ldots$ is an infinite ascending chain of ideals of R.

(g) $R = \begin{pmatrix} \mathbf{Z} & \mathbf{Q} \\ ٠ & \mathbf{Q} \end{pmatrix}$, R is not a left Noetherian ring since for positive integer, n, $A_n = \{ \begin{pmatrix} ٠ & \frac{m}{٢^n} \\ ٠ & ٠ \end{pmatrix} \mid m \in \mathbf{Z} \} \lhd_\ell R$ and $A_١ \subset A_٢ \subset \ldots$ is an infinite ascending chain of left ideals of R .

<u>Definition ١,١,٢٦</u>

A left R-module M is said to be a *left Artinian* if M satisfies the descending chain condition (d.c.c) on submodules of M i.e. for every descending chain of submodules of M, $N_١ \supseteq N_٢ \supseteq \ldots$,there exists a positive integer n such that $N_m = N_n$ for all $m \geq n$.

The ring R is a left Artinian if ${}_R R$ is a left Artinian module.

<u>Example ١,١,٢٧</u>

(a) Every field, and every finite ring is Artinian.

(b) $R = (\mathbf{Z}, +, .)$ is not Artinian ring because $\langle ٢ \rangle \supset \langle ٢^٢ \rangle \supset \ldots$ is an infinite descending chain of ideals of R.

<u>Definition ١,١,٢٨</u>

We say that the ring R is a semi-simple if $J(R) = ٠$ (see ٥, p ١٥٧).

Note that if R is semi-prime Artinian ,then it's semi-simple Artinian .

Definition ١,١,٢٩

(a) We say that R satisfies (a.c.c) on left annihilators if every ascending chain of left annihilators $I_١ \subseteq I_٢ \subseteq \ldots$ terminates after a finite number of steps.

(b) We say that R satisfies (a.c.c) on complement left ideals if every ascending chain of complement left ideals $K_١ \subseteq K_٢ \subseteq \ldots$ terminates after a finite number of steps.

(c) We say that R has no infinite direct sum of left ideals if every ascending chain of the form $I_١ \subseteq I_١ \oplus I_٢ \subseteq I_١ \oplus I_٢ \oplus I_٣ \subseteq \ldots$ terminates.

Note that :

R satisfies the a.c.c on complement left ideals if and only if R has no infinite direct sum of non-zero left ideals.

Definition ١,١,٣٠

A ring R is said to be a *left Goldie ring* if:

(a) R satisfies the a.c.c on left annihilator ideals.

(b) R satisfies the a.c.c on complement left ideals.

The right Goldie ring defined similarly.

Example ١,١,٣١

Every Noetherian ring is a Goldie ring and any integral domain is trivially a Goldie ring.

Definition ١,١,٣٢

A class Ӿ of rings is said to be :

(a) S-closed if S is a subring of R and $R \in$ Ӿ then $S \in$ Ӿ.

(b) I-closed if $J \lhd R$ and $R \in$ Ӿ then $J \in$ Ӿ .

(c) Q-closed if $R \in$ Ӿ and $I \lhd R$ then $R/I \in$ Ӿ .

(d) E-closed if $I \lhd R$ and I , $R/I \in$ Ӿ then $R \in$ Ӿ .

Note that

(a) The class of all Noetherian modules is S-closed , Q-closed and E-closed .

(b)The class of all left Noetherian rings is Q-closed and E-closed, but it is neither S-closed nor I-closed as the following example shows:

Example ١,١,٣٣

Let $S = F[x_{١}, x_{٢}, ...]$ then S is an integral domain , so it can be embedded in a field $(K, +, .)$ and $(K, +, .)$ is Noetherian , but $S \cong T \le K$ is not since $\langle x_{١} \rangle \subset \langle x_{١}, x_{٢} \rangle \subset \langle x_{١}, x_{٢}, x_{٣} \rangle \subset \ldots$ is an infinite ascending chain of ideals of S .

(b) Let $R = \begin{pmatrix} \mathbf{Q} & \mathbf{Q} \\ ٠ & \mathbf{Z} \end{pmatrix}$. Then R is a left Noetherian ring , but $I = \begin{pmatrix} ٠ & \mathbf{Q} \\ ٠ & \mathbf{Z} \end{pmatrix} \lhd R$, and I is not left Noetherian , since for each Positive

integer n , $A_n = \{ \begin{pmatrix} ٠ & \frac{m}{٢^n} \\ ٠ & ٠ \end{pmatrix} \mid m \in \mathbf{Z} \} \lhd_{\ell} I$, is an infinite ascending chain

$A_١ \subset A_٢ \subset \ldots$ of left ideals of I . Hence it is not I-closed .

<u>Theorem ١,١,٣٤</u>

Let R be a semi-prime ring with the a.c.c for right annihilators , then R has no non-zero nil one-side ideals .

<u>Theorem ١,١,٣٥</u>

Let R be a semi-prime ring with a.c.c on annihilator ideals . Then every annihilator ideal in R is the intersection of a finite number of minimal prime ideals of R .

CHAPTER TWO

Left Quasi-Noetherian Rings and Modules

This chapter consists of three sections, in the first we give definitions, examples and basic properties. In section two , we study internal structures and we give some classification of such classes of rings . In section three we study the relationship between the class of left Quasi-Noetherian ring and Goldie's Rings , Rings of Quotient and Polynomial Rings .

٢,١: Definitions and basic properties

In this part we define the left Quasi-Noetherian Rings and Modules , which is a generalization of left Noetherian Rings and Modules , it is also a generalization of nilpotent Rings, then we consider the problem of finding conditions which are equivalent to the definition . Finally we show that the class of left Quasi-Noetherian Modules is S-closed , Q-closed and E-closed while the class of left Quasi-Noetherian Rings is neither S-closed nor E-closed , but it is I-closed and Q-closed . We show also that a finite direct sum of left Quasi-Noetherian Rings is a left Quasi-Noetherian Ring.

Definition ٢,١,١

Let R be a ring , and let M be a left R-module , we say that M is a left Quasi-Noetherian if for every ascending chain $N_١ \subseteq N_٢ \subseteq \ldots \subseteq N_n \subseteq \ldots$ of R-submodules of M , there exists $m \in \mathbf{Z}^+$

such that $R^{m}(\bigcup_{n} N_{n}) \subseteq N_{m}$.

We say that the ring R is a left Quasi-Noetherian Ring if ${}_{R}R$ is a Quasi-Noetherian .

Examples ٢,١,٢

(a) Any left Noetherian Ring or Module is a left Quasi-Noetherian.

(b)Any nilpotent Ring is a left Quasi-Noetherian , hence

$R_{١} = \begin{pmatrix} ٠ & \mathbf{Q} \\ ٠ & ٠ \end{pmatrix} = (\{ \begin{pmatrix} ٠ & a \\ ٠ & ٠ \end{pmatrix} \mid a \in \mathbf{Q}\} , + , .)$, and

$R_{٢} = \begin{pmatrix} ٠ & ٠ \\ \mathbf{Q} & ٠ \end{pmatrix} = (\{ \begin{pmatrix} ٠ & ٠ \\ b & ٠ \end{pmatrix} \mid b \in \mathbf{Q} \} , + , .)$ are left Quasi-Noetherian since

$R_{١}^{٢} = ٠$ and $R_{٢}^{٢} = ٠$ but neither of them is left Noetherian . For if

$A_{n} = \{ \begin{pmatrix} ٠ & \frac{m}{٢^{n}} \\ ٠ & ٠ \end{pmatrix} \mid m \in \mathbf{Z} , n \in \mathbf{Z}^{+}\}$, then $A_{n} \lhd_{\ell} R_{١}$ for all n , and

$A_{١} \subsetneqq A_{٢} \subsetneqq \ldots$ is an infinite strictly ascending chain of left ideals of $R_{١}$,

and if $B_{n} = \{ \begin{pmatrix} ٠ & ٠ \\ \frac{m}{٢^{n}} & ٠ \end{pmatrix} \mid m \in \mathbf{Z}, n \in \mathbf{Z}^{+}\}$, then $B_{n} \lhd_{\ell} R_{٢}$ for all n , and

$B_{١} \subsetneqq B_{٢} \subsetneqq \ldots$ is an infinite strictly ascending chain of left ideals of $R_{٢}$.

(c) $R = \begin{pmatrix} \mathbf{Z} & ٠ \\ \mathbf{Q} & ٠ \end{pmatrix}$ is a non-nilpotent ring which is a left Quasi-Noetherian

since the ideals of R are of the form $\begin{pmatrix} 0 & 0 \\ \mathbf{Q} & 0 \end{pmatrix}$, and

$J_k = \{ \begin{pmatrix} 0 & 0 \\ r\,p^k & 0 \end{pmatrix} \mid r \in \mathbf{Z},\, k \in \mathbf{Z}^+,\, p \text{ is prime} \}$, and

$I_k = \{ \begin{pmatrix} 0 & 0 \\ \frac{r}{p^k} & 0 \end{pmatrix} \mid r \in \mathbf{Z},\, k \in \mathbf{Z}^+,\, P \text{ is prime} \}$. Therefore the ascending chains of left ideals in R are $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$, and there exists an $m = 1$ such that $R^m(\bigcup_n I_n) = R(\bigcup_n I_n) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \subseteq I_1$. Hence R is a left Quasi-Noetherian , but it is not a left Noetherian ring , for if inparticular $I_k = \{ \begin{pmatrix} 0 & 0 \\ \frac{r}{2^k} & 0 \end{pmatrix} \mid r \in \mathbf{Z},\, k \in \mathbf{Z}^+ \}$ then $I_k \lhd_\ell R$ for all k and $I_1 \subsetneqq I_2 \subsetneqq \ldots$ is an infinite strictly ascending chain of left ideals of R.

(d) Let $M = F[x]$, F is a field and $charF = 0$. Then M as an F-module is not a left Quasi-Noetherian , because if M is a left Quasi-Noetherian module then for any ascending chain $N_1 \subseteq N_2 \subseteq \ldots$ of F-submodule of M there exists $m \in \mathbf{Z}^+$ such that $F^m(\bigcup_n N_n) = F(\bigcup_n N_n) \subseteq N_m$. But $\bigcup_n N_n$ is a submodule of M, M unital , so $F(\bigcup_n N_n) = \bigcup_n N_n$, and $\bigcup_n N_n \subseteq N_m$, so $N_n \subseteq N_m$ for all $n \geq m$. But $N_m \subseteq N_n$. Therefore M is Noetherian which is a contradiction.

Lemma ٢,١,٣

(a) Let R be a ring ,and M be a left R-module. If $RM = ٠$, then M is

a left Quasi-Noetherian.

(b) Any left Quasi-Noetherian ring with identity is a left Noetherian.

Proof :

(a) Let $N_١ \subseteq N_٢ \subseteq ... \subseteq N_n \subseteq \ldots$ be any ascending chain of R-submodules of M then there exists an $m = ١ \in \mathbf{Z}^+$ such that $R^m(\bigcup_n N_n) = R(\bigcup_n N_n) \subseteq RM = ٠ \subseteq N_١$ so M is a left Quasi-Noetherian module.

(b) Let $I_١ \subseteq I_٢ \subseteq ... \subseteq I_n \subseteq \ldots$ be any ascending chain of left ideals of R, since R is a left Quasi-Noetherian so there exists $m \in \mathbf{Z}^+$ such that $R^m(\bigcup_n I_n) = R(\bigcup_n I_n) \subseteq I_m$ but R with identity and $\bigcup_n I_n \lhd_\ell R$ so $R(\bigcup_n I_n) = \bigcup_n I_n$ then $\bigcup_n I_n \subseteq I_m$ i.e. $I_m = I_{m+١} = \ldots$.

Hence the chain terminates so R is a left Noetherian .

□

From lemma ٢,١,٣, we have the following

(a) $R = (F[x_١, x_٢, ..., x_n, ...], +, .)$ is not a Quasi-Noetherian ring .

(b) $R=\begin{pmatrix} \mathbf{Z} & \mathbf{Q} \\ ٠ & \mathbf{Q} \end{pmatrix}$ is not a left Quasi-Noetherian ring .

Now we prove the following characterization of a left Quasi-Noetherian Modules .

<u>Theorem ٢,١,٤</u>

Let M be a left R-module. Then the following conditions are equivalent:

(c) M is a left Quasi-Noetherian.

(d) In each non-empty collection Γ of R-submodules of M, there is an element N and an $m \in \mathbf{Z}^+$ such that $R^m K \subseteq N$ for every $K \in \Gamma$ and $K \supseteq N$.

(c) In each non-empty collection Γ of R-submodules of M such that $K \in \Gamma \Rightarrow K :_M R = \{m \in M \mid Rm \subseteq K\} \in \Gamma$, there exists a maximal element.

(d) For every ascending chain of R-submodules $N_١ \subseteq N_٢ \subseteq \ldots \subseteq N_n \subseteq \ldots$ there exists $m \in \mathbf{Z}^+$ such that the ascending chain $N_١ :_M R^m \subseteq N_٢ :_M R^m \subseteq \ldots \subseteq N_n :_M R^m \subseteq \ldots$ of R-submodules is terminate.

<u>Proof</u> :

(a)$\Rightarrow$(b) Let M be a left Quasi-Noetherian module . Let Γ be a non-empty collection of R-submodules of M. Suppose that for all $N \in \Gamma$ and $m \in \mathbf{Z}^+$, there exists $K \in \Gamma$ such that $K \supseteq N$, but $R^m K \not\subseteq N$. Pick

an R-submodule $N_1 \in \Gamma$ then there exists $N_2 \in \Gamma$, $N_1 \subseteq N_2$ such that $RN_2 \not\subseteq N_1$, like wise there exists $N_3 \in \Gamma$, $N_2 \subseteq N_3$ such that $R^2 N_3 \not\subseteq N_2$, continuing in this way, we obtain an ascending chain $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ of R-submodules of M such that $R^n N_{n+1} \not\subseteq N_n$, for all $n \in \mathbf{Z}^+$, therefore $R^m(\bigcup_n N_n) \not\subseteq N_m$ for all $m \in \mathbf{Z}^+$ which contradicts the hypothesis that M is a left Quasi-Noetherian module.

(b) $\Rightarrow$(c) Let Γ be a non-empty collection of R-submodules of M such that $K :_M R \in \Gamma$ for all $K \in \Gamma$. Then $K :_M R^n \in \Gamma$ for all $n \in \mathbf{Z}^+$. But $K \subseteq K :_M R^n$ for all $n \in \mathbf{Z}^+$, hence by (b) there exists an $m \in \mathbf{Z}^+$ such that $R^m(K :_M R^n) \subseteq K$ for all $n \in \mathbf{Z}^+$. Therefore $K :_M R^n \subseteq K :_M R^m$ for all $n \in \mathbf{Z}^+$. Hence $K :_M R^n = K :_M R^m$ for all $n \geq m$, and Γ has a maximal element.

(c) $\Rightarrow$(d) Let $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ be any ascending chain of R-submodules and suppose that the ascending chain $N_1 :_M R^m \subseteq N_2 :_M R^m \subseteq \ldots \subseteq N_n :_M R^m \subseteq \ldots$ of R-submodules does not terminate for all $m \in \mathbf{Z}^+$. Therefore the collection $\Gamma = \{ N_1, \ldots, N_1 :_M R, \ldots, N_1 :_M R^m, \ldots \}$ is a non-empty collection of R-submodules and for all $N \in \Gamma$ we have $N :_M R \in \Gamma$ but Γ has no maximal element, which is a contradiction.

(d) $\Rightarrow$(a) Let $N_1 \subseteq N_2 \subseteq ... \subseteq N_n \subseteq ...$ be any ascending chain of R-submodules of M then there exists $m \in \mathbf{Z}^+$ such that $N_1 :_M R^m \subseteq N_2 :_M R^m \subseteq ... \subseteq N_n :_M R^m \subseteq ...$ forms an ascending chain of R-submodules of M and by (d) there exists $s \in \mathbf{Z}^+$ such that $N_s :_M R^m = N_n :_M R^m$ for all $n \geq s$ then $N_s :_M R^m = \bigcup_n (N_n :_M R^m)$ so $R^m(\bigcup_n (N_n :_M R^m)) \subseteq N_s$ then $R^m(\bigcup_n N_n) \subseteq N_s$. Take $t = \max\{m, s\}$ then $R^t(\bigcup_n N_n) \subseteq N_t$ hence M is a left Quasi-Noetherian module .

□

Corollary ٢,١,٥

Let R be a ring. Then the following conditions are equivalent:

(a) R is a left Quasi-Noetherian ring .

(b) In each non-empty collection Γ of left ideals of R, there is an element I and an $m \in \mathbf{Z}^+$ such that $R^m J \subseteq I$ for every $J \in \Gamma$ and $J \supseteq I$.

(c) In each non-empty collection Γ of left ideals of R such that $J \in \Gamma \Rightarrow J : R = \{x \in R \mid Rx \subseteq J\} \in \Gamma$, there exists a maximal element.

(d) For every ascending chain $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \subseteq ...$ of left ideals of R there exists $m \in \mathbf{Z}^+$ such that the ascending chain $I_1 : R^m \subseteq I_2 : R^m \subseteq ... \subseteq I_n : R^m \subseteq ...$ of left ideals of R is terminate .

Proof :

Regarded R as R-module, then the result will follow. □

Theorem ٢,١,٦

Let M be a left R-module. If RM is a left Noetherian, then M is a left Quasi-Noetherian.

Proof :

Let $A_١ \subseteq A_٢ \subseteq ... \subseteq A_n \subseteq ...$ be any ascending chain of R-submodules of M. Then $RA_١ \subseteq RA_٢ \subseteq ... \subseteq RA_n \subseteq ...$ is an ascending chain of R-submodules of RM. But RM is a left Noetherian, hence $RA_r = RA_s \subseteq A_s$ for all $r \geq s$. Therefore $R(\bigcup_n A_n) \subseteq A_s$ and M is a left Quasi-Noetherian.

□

Remark ٢,١,٧

The converse of Theorem (٢,١,٦) is not necessarily true as the following example shows :

Let $R = \{\begin{pmatrix} ٠ & ٠ \\ \frac{r}{p^k} & ٠ \end{pmatrix} \mid r \in \mathbf{Z}, k \in \mathbf{Z}^+, p \text{ is prime}\}$ and

$M = \{\begin{pmatrix} a & ٠ \\ b & ٠ \end{pmatrix} \mid a \in \mathbf{Z}, b \in \mathbf{Q}\}$ so M is a left R-module , and for every ascending chain $N_١ \subseteq N_٢ \subseteq ... \subseteq N_n \subseteq ...$ of R-submodules of M there exists $m = ٢ \in \mathbf{Z}^+$ such that $R^m(\bigcup_n N_n) = R^٢(\bigcup_n N_n) = ٠ \subseteq N_١$. Hence M is a left Quasi-Noetherian module . But

$$RM = \{ \begin{pmatrix} \cdot & \cdot \\ \frac{t}{p^k} & \cdot \end{pmatrix} | t \in \mathbf{Z}, k \in \mathbf{Z}^+, p \ is \ prime \} = R$$, is not a left Noetherian .

Now let $\mathfrak{X}$ be the class of all left Quasi-Noetherian Rings and let $\mathfrak{M}$ be the class of all left Quasi-Noetherian Modules . Therefore if A is a submodule of N and N is a submodule of M, then clearly A is a submodule of M. Hence if M is a left Quasi-Noetherian and N is an R-submodule of M , then N is a left Quasi-Noetherian and $\mathfrak{M}$ is an S-closed.

Next we prove the following

Theorem ٢,١,٨

(a) $\mathfrak{M}$ is Q-closed.

(b) $\mathfrak{X}$ is Q-closed.

Proof :

(a) Suppose that $M \in \mathfrak{M}$ and let N be an R-submodule of M and let $\pi : M \to M/N = \overline{M}$ be the natural homomorphism . Now if $\overline{N}_{١} \subseteq \overline{N}_{٢} \subseteq ... \subseteq \overline{N}_n \subseteq ...$ be an ascending chain of R-submodules of $\overline{M}$. Put $N_k = \pi^{-١}(\overline{N}_k)$, $k = ١, ٢, ...$. Then $N_{١} \subseteq N_{٢} \subseteq ... \subseteq N_n \subseteq ...$ is an ascending chain of R-submodules of M, but M is a left Quasi-Noetherian hence there exists an $m \in \mathbf{Z}^+$ such that $R^m(\bigcup_n N_n) \subseteq N_m$. Therefore $\pi(R^m(\bigcup_n N_n)) \subseteq \pi(N_m)$. Hence $R^m\pi(\bigcup_n N_n) \subseteq \pi(N_m)$ and

$R^{m}(\bigcup_{n}\overline{N}_{n})\subseteq\overline{N}_{m}$. Therefore $\overline{M}$ is a left Quasi-Noetherian.

The proof of (b) is similar.

□

The converse of part (a) of Theorem ٢,١,٨ is also true , as the following Theorem shows .

Theorem ٢,١,٩

If $M/N\in\mathcal{M}$ for all $٠\neq N\leq M$, then $M\in\mathcal{M}$.

Proof :

Let $A_{١}\subseteq A_{٢}\subseteq\ldots\subseteq A_{n}\subseteq\ldots$ be any ascending chain of R- submodules of M. Then $A_{٢}/A_{١}\subseteq A_{٣}/A_{١}\subseteq\ldots\subseteq A_{n}/A_{١}\subseteq\ldots$ is an ascending chain of R-submodules of $M/A_{١}$. But $M/A_{١}$ is a left Quasi-Noetherian module, hence there exists $m\in\mathbf{Z}^{+}$ such that $R^{m}(\bigcup_{n}A_{n}/A_{١})\subseteq A_{m}/A_{١}$. Therefore $R^{m}(\bigcup_{n}A_{n})\subseteq A_{m}$. Hence M is a left Quasi-Noetherian module.

□

Remark ٢,١,١٠

The converse of part (b) of Theorem ٢,١,٨ is not true as the following example shows :

Let $R=\begin{bmatrix}\mathbf{Z} & \mathbf{Q}\\ ٠ & \mathbf{Q}\end{bmatrix}$. Then the ideals of R are of the form

$I_{١}=\begin{bmatrix} \mathbf{Z} & \mathbf{Q} \\ ٠ & ٠ \end{bmatrix}$, $I_{٢}=\begin{bmatrix} ٠ & \mathbf{Q} \\ ٠ & \mathbf{Q} \end{bmatrix}$, $I_{٣}=\begin{bmatrix} ٠ & \mathbf{Q} \\ ٠ & ٠ \end{bmatrix}$ and

$R/I_{١}\cong\begin{bmatrix} ٠ & ٠ \\ ٠ & \mathbf{Q} \end{bmatrix}\cong\mathbf{Q}$, $R/I_{٢}\cong\begin{bmatrix} \mathbf{Z} & ٠ \\ ٠ & ٠ \end{bmatrix}\cong\mathbf{Z}$, $R/I_{٣}\cong\begin{bmatrix} \mathbf{Z} & ٠ \\ ٠ & \mathbf{Q} \end{bmatrix}\cong\mathbf{Z}\oplus\mathbf{Q}$ are left Quasi-Noetherian rings . Hence R/I is a left Quasi-Noetherian for all $٠\neq I\lhd R$ but R is not , however we have the following

Theorem ٢,١,١١

Let R be a commutative ring . Then if R/I is a Quasi-Noetherian ring for all $٠\neq I\lhd R$, then R is a Quasi-Noetherian.

Proof :

Let $I_{١}\subseteq I_{٢}\subseteq\ldots\subseteq I_n\subseteq\ldots$ be any ascending chain of ideals of R. Then $I_{٢}/I_{١}\subseteq I_{٣}/I_{١}\subseteq\ldots\subseteq I_n/I_{١}\subseteq\ldots$ is an ascending chain of ideals of $R/I_{١}$. But $R/I_{١}$ is a Quasi-Noetherian ring, hence there exists $m\in\mathbf{Z}^{+}$ such that $(R/I_{١})^m(\bigcup_n I_n/I_{١})\subseteq I_m/I_{١}$, then

$$\pi^{-١}((R/I_{١})^m)\pi^{-١}(\bigcup_n I_n/I_{١})\subseteq\pi^{-١}((R/I_{١})^m(\bigcup_n I_n/I_{١}))\subseteq\pi^{-١}(I_m/I_{١}) .$$

Therefore $R^m(\bigcup_n I_n)\subseteq I_m$. Hence R is a Quasi-Noetherian ring .

□

Next we prove the following

Theorem ٢,١,١٢

$\mathcal{QN}$ is E-closed.

Proof :

Let N be an R-submodule of M, and suppose that N and M/N in $\mathcal{T}$. Let $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ be any ascending chain of R-submodules of M. Then $N_1 \cap N \subseteq N_2 \cap N \subseteq \ldots \subseteq N_n \cap N \subseteq \ldots$ is an ascending chain of R-submodules of N and $\frac{N_1+N}{N} \subseteq \frac{N_2+N}{N} \subseteq \ldots \subseteq \frac{N_n+N}{N} \subseteq \ldots$ is an ascending chain of R-submodules of M/N. But N and M/N in $\mathcal{T}$, hence there exists $t \in \mathbf{Z}^+$ such that $R^t(\bigcup_n N_n \cap N) \subseteq N_t \cap N$, which implies that $R^t((\bigcup_n N_n) \cap N) \subseteq N_t \cap N$. Also there exists $s \in \mathbf{Z}^+$ such that $R^s(\bigcup_n \frac{N_n+N}{N}) \subseteq \frac{N_s+N}{N}$, which implies that $R^s(\bigcup_n N_n + N) \subseteq N_s + N$.

Take $m = \max\{t, s\}$ then $R^m(\bigcup_n N_n \cap N) \subseteq N_m \cap N$ ...(1)

and $R^m(\bigcup_n N_n + N) \subseteq N_m + N$ ...(2)

Now $R^m(\bigcup_n N_n) \subseteq R^m(\bigcup_n N_n + N) \cap (\bigcup_n N_n)$

$\subseteq (N_m + N) \cap (\bigcup_n N_n)$ ... By (2)

$= (\bigcup_n N_n \cap N) + N_m$ By modular law

so $R^m(R^m(\bigcup_n N_n)) \subseteq R^m((\bigcup_n N_n \cap N) + N_m)$

$= R^m(\bigcup_n N_n \cap N) + R^m N_m$

$\subseteq (N_m \cap N) + N_m = N_m$. ...By (1)

Therefore $R^{٢m}(\bigcup_{n} N_n) \subseteq N_m \subseteq N_{٢m}$. Hence M is a left Quasi-Noetherian module.

□

<u>Corollary ٢,١,١٣</u>

A finite direct sum of left Quasi-Noetherian modules is a left Quasi-Noetherian.

□

<u>Remark ٢,١,١٤</u>

$\mathfrak{X}$ is not S-closed , as the following example shows:

Let $S = F[x_١, x_٢, ..., x_n, ...]$ be the infinite polynomial ring over the field of characteristic zero F, then S is an integral domain , and S can be imbedded in a field K. Hence K is a left Quasi-Noetherian and $S \leq K$, but S is not a left Quasi-Noetherian . Otherwise S is a left Noetherian which impossible since $\langle x_١ \rangle \subsetneqq \langle x_١, x_٢ \rangle \subsetneqq ...$ is an infinite strictly ascending chain of ideals of S, however we have the following

<u>Theorem ٢,١,١٥</u>

$\mathfrak{X}$ is I-closed .

<u>Proof</u> :

Let R be a left Quasi-Noetherian ring , $I \lhd_{\ell} R$, and let $J_١ \subseteq J_٢ \subseteq ... \subseteq J_n \subseteq ...$ be any ascending chain of left ideals of I, then

$IJ_{١} \subseteq IJ_{٢} \subseteq \ldots \subseteq IJ_n \subseteq \ldots$ is an ascending chain of left ideals of R . But R is a left Quasi-Noetherian ring so there exists $m \in \mathbf{Z}^{+}$ such that $R^m(\bigcup_n IJ_n) \subseteq IJ_m \subseteq J_m$. Since $I^m \subseteq R^m$ then $I^m(\bigcup_n IJ_n) = I^{m+١}(\bigcup_n J_n) \subseteq J_m \subseteq J_{m+١}$. Hence I is a left Quasi-Noetherian.

□

<u>Remark ٢,١,١٦</u>

$\mathfrak{X}$ is not E-closed , as the following example shows :

Let $R = \begin{pmatrix} \mathbf{Z} & \mathbf{Q} \\ ٠ & \mathbf{Q} \end{pmatrix}$, $I = \begin{pmatrix} ٠ & \mathbf{Q} \\ ٠ & ٠ \end{pmatrix}$, then $I \lhd R$, and $I^{٢} = ٠$ hence $I \in \mathfrak{X}$.

But $R/I \cong \mathbf{Z} \oplus \mathbf{Q}$ which is a left Noetherian , therefore $R/I \in \mathfrak{X}$. But $R \notin \mathfrak{X}$, by example (b) p (١٧) , however we have the following

<u>Theorem ٢,١,١٧</u>

A finite direct sum of left Quasi-Noetherian rings is a left Quasi-Noetherian.

<u>Proof</u> :

It is enough to prove the result for $t = ٢$. So let $R = R_{١} \oplus R_{٢}$, $R_{١}$, $R_{٢}$ are left Quasi-Noetherian . Now let $I_{١} \subseteq I_{٢} \subseteq \ldots \subseteq I_n \subseteq \ldots$ be any ascending chain of left ideals of R . Then $R_{١}I_{١} \subseteq R_{١}I_{٢} \subseteq \ldots \subseteq R_{١}I_n \subseteq \ldots$ is an ascending chain of left ideals of $R_{١}$, and $R_{٢}I_{١} \subseteq R_{٢}I_{٢} \subseteq \ldots \subseteq R_{٢}I_n \subseteq \ldots$

is an ascending chain of left ideals of R_2. But R_1 and R_2 are left Quasi-Noetherian rings , therefore there exist $r, s \in \mathbf{Z}^+$ such that

$R_1^r(\bigcup_n R_1 I_n) \subseteq R_1 I_r \subseteq I_r$ and $R_2^s(\bigcup_n R_2 I_n) \subseteq R_2 I_s \subseteq I_s$. Now let $m = \max\{r, s\}$, then $R_1^m(\bigcup_n R_1 I_n) \subseteq R_1 I_m \subseteq I_m$ and $R_2^m(\bigcup_n R_2 I_n) \subseteq R_2 I_m \subseteq I_m$. Hence

$R^{m+1}(\bigcup_n I_n) \subseteq R_1^m(\bigcup_n R_1 I_n) + R_2^m(\bigcup_n R_2 I_n) \subseteq I_m \subseteq I_{m+1}$.

Therefore R is a left Quasi-Noetherian ring .

□

Note that the converse of Theorem ٢,١,١٧ also hold for if

$R = R_1 \oplus R_2$

and R is a left Quasi-Noetherian then by Theorem

٢,١,١٥ $R_1 \cong R_1 \oplus \{0\} \lhd R$ and $R_2 \cong \{0\} \oplus R_2 \lhd R$ are left Quasi-Noetherian.

Finally we prove the following

Theorem ٢,١,١٨

If $R \in$ Ӿ, then R satisfies the ascending chain condition on semi-prime ideals.

Proof :

Let $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$ be any ascending chain of semi-prime ideals

of R. Since R is a left Quasi-Noetherian ring then there exists $m \in \mathbf{Z}^+$ such that $R^m(\bigcup_n I_n) \subseteq I_m$. But $\bigcup_n I_n \lhd R$, hence $(\bigcup_n I_n)^m \subseteq R^m$ and $(\bigcup_n I_n)^{m+1} = (\bigcup_n I_n)^m(\bigcup_n I_n) \subseteq R^m(\bigcup_n I_n) \subseteq I_m$. But I_m is a semi-prime ideal, hence $(\bigcup_n I_n) \subseteq I_m$ i.e. $I_m = I_{m+1} = \dots$.

□

<u>Corollary ٢,١,١٩</u>

If $R \in \mathfrak{X}$, then R satisfies the ascending chain condition on prime ideals.

<u>Proof</u> :

Since every prime ideal is semi-prime ideal, the result will follow.

□

<u>٢,٢: The ideal structure and some classification</u>

In this section we study the structure of ideals in a left Quasi-Noetherian ring.

First we prove the following

<u>Theorem ٢,٢,١</u>

If $R \in \mathfrak{X}$, then every nil left ideal in R is nilpotent .

To prove this we need the following lemma.

<u>Lemma ٢,٢,٢</u>

Let R be a semi-prime left Quasi-Noetherian .Then R satisfies the ascending chain condition on annihilator ideals .

<u>Proof</u> :

Let $I_١ \subseteq I_٢ \subseteq \ldots \subseteq I_n \subseteq \ldots$ be any ascending chain of annihilator ideals of R, since R is a left Quasi-Noetherian, then there exists $m \in \mathbf{Z}^+$ such that $R^m(\bigcup_n I_n) \subseteq I_m$. Therefore

$\bigcup_n I_n \subseteq I_m : R^m = ٠ : J_m : R^m = ٠ : J_m R^m = ann\, J_m R^m$.

Thus $\bigcup_n I_n (J_m R^m) = ٠$, and $((\bigcup_n I_n)\, J_m)\, R^m = ٠$. But $((\bigcup_n I_n)\, J_m) \subseteq R$, hence $((\bigcup_n I_n)\, J_m)^{m+١} = ٠$. Since R is a semi-prime it follows that $(\bigcup_n I_n)\, J_m = ٠$, and $\bigcup_n I_n \subseteq ann\, J_m = I_m$. Hence $\bigcup_n I_n = I_m$ and $I_n \subseteq I_m$, but $I_n \supseteq I_m$, hence $I_n = I_m$ for all $n \geq m$.

□

<u>Proof of theorem ٢,٢,١</u>

Let $\{I_١ , I_٢ , \ldots \}$ be a set of all nilpotent left ideals of R . Then $I_١ \subseteq I_١ + I_٢ \subseteq \ldots \subseteq I_١ + I_٢ + \ldots + I_n \subseteq \ldots$ is an ascending chain of left ideals of R . It's clear that the union of all left ideals in the chain is $\sum_{i=١} I_i$. Since R is a left Quasi-Noetherian ring , then there exists $m \in \mathbf{Z}^+$ such that

$R^m(\sum_{i=١} I_i) \subseteq \sum_{i=١}^{m} I_i$. But $(\sum_{i=١} I_i)^m \subseteq R^m$ and $(\sum_{i=١} I_i)^{m+١} \subseteq \sum_{i=١}^{m} I_i$. But $\sum_{i=١}^{m} I_i$ is a finite sum of nilpotent left ideals, hence $\sum_{i=١}^{m} I_i$ is a nilpotent left ideal of R. Therefore there exists $t \in \mathbf{Z}^+$ such that $(\sum_{i=١}^{m} I_i)^t = ٠$. Hence $((\sum_{i=١} I_i)^{m+١})^t = (\sum_{i=١} I_i)^{m+t+١} = ٠$ and $\sum_{i=١} I_i$ is a nilpotent ideal of R .

Now let $W = \sum_{i=١} I_i$ then R/W has no non-zero nilpotent ideals , and R/W is a semi-prime left Quasi-Noetherian ring. Therefore by Lemma ٢,٢,٢ R/W satisfies the ascending chain condition on (left , right) annihilator ideals. Hence by Theorem ١,١,٣٤ R/W has no non-zero nil left ideals. Now let I be a nil left ideal of R , and $\pi: R \to R/W$ be the natural homomorphism . Then $\bar{I} = \pi(I) = \bar{٠}$ and $I \subseteq W$. Hence I is nilpotent .

□

An immediate consequence of Theorem ٢,٢,١ we have the following

Corollary ٢,٢,٣

Let $N(R)$ be the nil radical of R. If $R \in \mathfrak{X}$, then $N(R)$ is nilpotent.

□

Next we have the following , which classify the class of left Quasi-Noetherian rings.

Theorem ٢,٢,٤

$R \in \mathfrak{X}$ if and only if $N = N(R)$ is nilpotent and each of the R-modules $R/N,\ N/N^{٢},\ N^{٢}/N^{٣}, \ldots$ is a left Quasi-Noetherian.

Proof :

Suppose that $R \in \mathfrak{X}$. Then by Corollary ٢,٢,٣ N is nilpotent . Now

${}_R M = {}_R R \in \mathcal{M}$, and $N^i \lhd R$ for all i , therefore $N^i \leq M$ for all i .But $\mathcal{M}$ is Q-closed , hence R/N^i is a left Quasi-Noetherian R-module for all i. Also $N^i/N^{i+١}$ is an R-submodule of $R/N^{i+١}$ for all $i \geq ١$ so each $N^i/N^{i+١}$ is a left Quasi-Noetherian R-module .

To prove the converse

Since $R/N \cong R/N^{٢} / N/N^{٢}$ and R/N $, N/N^{٢}$ are left Quasi-Noetherian R-module , it follows from Theorem (٢,١,١٢) , that $R/N^{٢} \in \mathcal{M}$, and by induction , $R/N^i \in \mathcal{M}$ for all i . But N is nilpotent , hence there exists

$m \in \mathbf{Z}^+$ such that $N^m = ٠$. Therefore $R \cong R/N^m$ is a left Quasi-Noetherian R-module . Hence R is a left Quasi-Noetherian ring .

□

Next we prove the following

Theorem ٢,٢,٥

If R is a commutative Quasi-Noetherian ring , then every irreducible ideal is a primary .

Proof :

Let $I \triangleleft R$ be an irreducible ideal . We need to show that I is a primary , since the homomorphic image of a Quasi-Noetherian Ring is a Quasi-Noetherian , it is enough to show that ٠ is a primary ideal . It sufficient to show that if $xy = ٠ \in R$, then either $x = ٠$ or $y^n = ٠$ for some n.

First we claim that $\langle x \rangle \bigcap \langle y^n \rangle = ٠$.

To prove this , we consider the ascending chain $ann\langle y \rangle \subseteq ann\langle y^٢ \rangle \subseteq ... \subseteq ann\langle y^n \rangle \subseteq ...$ of ideals of R , since R is a Quasi-Noetherian ring , then there exists $m \in \mathbf{Z}^+$ such that $R^m(\bigcup_n ann\langle y^n \rangle) \subseteq ann\langle y^m \rangle = ٠ : \langle y^m \rangle$ so

$\bigcup_n (٠ : \langle y^n \rangle) \subseteq ٠ : \langle y^m \rangle R^m \subseteq ٠ : \langle y^{٢m} \rangle$ so $\bigcup_n ann \langle y^n \rangle = ann \langle y^{٢m} \rangle$

and the chain terminates . Therefore $ann \langle y^n \rangle = ann \langle y^{n+١} \rangle$. Now if $a \in \langle x \rangle \bigcap \langle y^n \rangle$, then $a = bx = cy^n$ for some b, c so $ay = ٠$ since $(bx)y = b(xy) = ٠$, it follows that $cy^n y = cy^{n+١} = ٠$, and $c \in ann\langle y^{n+١} \rangle = ann \langle y^n \rangle$. Therefore $a = cy^n = ٠$ and $\langle x \rangle \bigcap \langle y^n \rangle = ٠$.

Now by irreducibility of the zero ideal , it follows that $\langle x \rangle = ٠$ or $\langle y^n \rangle = ٠$. Hence $x = ٠$ or $y^n = ٠$.

□

Now we raise the following question .

Let R be a commutative Quasi-Noetherian ring . Is every ideal of R is a finite intersection of primary ideals ?

We know no example disproving this result , however we have the following .

Theorem ٢,٢,٦

Let R be a commutative Quasi-Noetherian ring , then every ideal of R is contained in a finite intersection of primary ideals .

To prove this we need the following lemma

Lemma ٢,٢,٧

Let R be a commutative Quasi-Noetherian ring , then every ideal of R is contained in a finite intersection of irreducible ideals .

Proof :

Since $I \subseteq I : R^t$ for all $t \in \mathbf{Z}^+$. It is enough to show that $I : R^t$ is contained in a finite intersection of irreducible ideals. Suppose that $I : R^t$ is a finite intersection of irreducible ideals so nothing to prove . If it is not , then it is not irreducible, therefore $I : R^t = A_{١} \cap B_{١}$, where $I : R^t \subset A_{١}$ and

$I : R^t \subset B_1$ but $A_1 \subseteq A_1 : R^t$ and $B_1 \subseteq B_1 : R^t$ If both $A_1 : R^t$ and $B_1 : R^t$ are irreducible then we are done . Otherwise we may assume that $A_1 : R^t$ is reducible . Continuing in this way , we get an ascending chain $I : R^t \subset A_1 : R^t \subset A_2 : R^t \subset \ldots$ of ideals of R. Since R is a Quasi-Noetherian ring , this chain must be terminate (by Theorem ٢,١,٥) and hence we are done .

□

Proof of theorem ٢,٢,٦ :

The result will follow from Theorem ٢,٢,٥ and Lemma ٢,٢,٧ .

□

Next we prove the following which prove the existence of the prime ideal in the class of left Quasi-Noetherian ring .

Theorem ٢,٢,٨

Let R be a non-nilpotent left Quasi-Noetherian ring, then there exists a prime ideal in R which is a right annihilator of some non-zero ideal of R.

Proof :

By Theorem ٢,٢,١ R is not nil, so R contains at least one non-nilpotent

element a (say), therefore the set $S = \{ a, a^2, \ldots, a^n, \ldots \}$ is closed under multiplication and doesn't contain zero. Thus there exists a prime ideal P of

R and $P \bigcap S = \Phi$ by Corollary of [٥, p ١٦٤].

Now let $a \in I_١ \lhd R$ so $I_١ r(I_١) = ٠ \subseteq P$ and $I_١ \not\subseteq P$ (since $a \in S$) , it follows that $r(I_١) \subseteq P$, if $P = r(I_١)$ then we finish , if not we can choose $x_١ \in R$ with $x_١ \notin P$ so if $x_١ \in I_٢ \lhd R$, then $I_١ I_٢ \not\subseteq P$ but then $r(I_١ I_٢) \subseteq P$. Continuing in this way we have $r(I_١) \subseteq r(I_١ I_٢) \subseteq ... \subseteq r(I_١ ... I_n) \subseteq ...$ (١) which is an ascending chain of right annihilator ideals of R, hence $٠ :_\ell I_١ \subseteq ٠ :_\ell I_١ I_٢ \subseteq ... \subseteq ٠ :_\ell I_١ ... I_n \subseteq ...$. But R is a left Quasi-Noetherian ring, hence there exists $m \in \mathbf{Z}^+$ such that $R^m (\bigcup_n ٠ :_\ell I_١ ... I_n) \subseteq ٠ :_\ell I_١ ... I_m$.

Therefore $\bigcup_n (٠ :_\ell I_١ ... I_n) \subseteq ٠ :_\ell I_١ ... I_m R^m$ and

$\bigcup_n (٠ :_\ell I_١ ... I_n) \subseteq ٠ :_\ell I_١ ... I_m I_{m+١} ... I_{٢m}$ which means that (١) terminates , i.e. $r(I_١ ... I_{٢m}) = r(I_١ ... I_{٢m+١}) = ...$. But $r(I_١ ... I_{٢m}) \subseteq P$, hence $\ell(r(I_١ ... I_{٢m})) \supseteq \ell(P)$, and $r(I_١ ... I_{٢m}) = r(\ell(r(I_١ ... I_{٢m}))) \subseteq r(\ell(P))$.

By maximalty of $r(I_١ ... I_{٢m})$ we have $P \subseteq r(\ell(P)) = r(I_١ ... I_{٢m})$.

Therefore $r(I_١ ... I_{٢m}) = P$ and P is a right annihilator of some non-zero ideal of R . □

Theorem ٢,٢,٩

Let $R \in \mathfrak{X}$, then every ideal of R contains a product of prime ideals .

Proof :

Suppose that the family F of ideals of R which doesn't contains

any product of prime ideals is non-empty .Therefore in this family we have for all $I \in F$ implies that $I :_\ell R \in F$ [Because if $\prod_{i=١}^{n} P_i \subseteq I :_\ell R$, for all $P_i \supseteq I :_\ell R \supseteq I$, then $R(\prod_{i=١}^{n} P_i) \subseteq I$ so $(\prod_{i=١}^{n} P_i)^{٢} \subseteq I$ for all $P_i \supseteq I$]. Since $R \in \mathfrak{X}$, then F has a maximal element (say) I . But I cannot be a prime ideal , hence there are ideals A and B of R such that $AB \subseteq I$ but $A \not\subseteq I$ and $B \not\subseteq I$. Consider $(I+A)(I+B) \subseteq I^{٢} + IB + AI + AB \subseteq I$ since $I \subsetneqq I+A$, $I \subsetneqq I+B$ then $I+A \notin F$, $I+B \notin F$. Thus $I+A$, $I+B$ each contain a product of prime ideals and hence so also does I. This is a contradiction . Hence $F = \Phi$

□

Corollary ٢,٢,١٠

Let $R \in \mathfrak{X}$. Then :

(a) There exists a finite number of prime ideals $P_١,...,P_n$ such that $\prod_{i=١}^{n} P_i = ٠$.

(b) every ideal of R contains a power of it's prime radical .

Proof :

(a) It's clear .

(b) Since $R\in$ Ӿ, $I \lhd R$, then by Theorem ٢,٢,٩ $\prod_{i=١}^{n} P_i \subseteq I$.

But

$rad(I)\subseteq P_i$ for all i , then $(rad(I))^n \subseteq \prod_{i=١}^{n} P_i$. Hence $(rad(I))^n \subseteq I$.

□

٣.Goldie's Rings, Rings of left Quotient and Polynomial Rings.

In this section we will study the relationship between left Quasi-Noetherian rings and some certain special types of rings (most notably Goldie rings , ring of left quotient and polynomial ring) .

We begin with the following.

Remark ٢,٣,١

If R is a left Quasi-Noetherian ring , then R need not be a left Goldie ring as the following examples shows:

(a) Let $R=\begin{pmatrix} ٠ & ٠ \\ \mathbf{Q} & ٠ \end{pmatrix}$. Then $R^{٢} = ٠$, hence R is a left Quasi-Noetherian ring , but it is not a left Goldie ring for if $I_n = \{ \begin{pmatrix} ٠ & ٠ \\ \frac{m}{٢^n} & ٠ \end{pmatrix} \mid m\in \mathbf{Z}, n\in \mathbf{Z}^+ \}$, then I_n is a left annihilator ideal in R for all n , and $I_١ \subsetneq I_٢ \subsetneq \ldots$ is an infinite strictly ascending chain of left annihilator ideals in R.

Note that $r(R) \neq ٠$, the right annihilator of R.

(b)Let $R = \begin{pmatrix} \mathbf{Z} & ٠ \\ \mathbf{Q} & ٠ \end{pmatrix}$.Then R is a left Quasi-Noetherian ring (see example ١,١,٢(c)) but it's not a left Goldie ring for if

$$I_n = \{ \begin{pmatrix} ٠ & ٠ \\ \frac{m}{٢^n} & ٠ \end{pmatrix} | m \in \mathbf{Z}, n \in \mathbf{Z}^+ \},$$

then I_n is a left annihilator ideal in R for all n, and $I_١ \subsetneqq I_٢ \subsetneqq \ldots$ is an infinite strictly ascending chain of left annihilator ideals in R.

Note that : $r(R) = \begin{pmatrix} ٠ & ٠ \\ \mathbf{Q} & ٠ \end{pmatrix} \neq ٠$, but $\ell(R) = ٠$.

However we prove the following which gives the relationship between left Quasi-Noetherian rings and left Goldie rings.

Theorem ٢,٣,٢

Let $R \in \mathfrak{X}$ and $r(R) = ٠$, then R is a left Goldie ring .

Proof :

First we show that any ascending chain of left annihilator ideals terminates . Let $J_١ \subseteq J_٢ \subseteq \ldots \subseteq J_n \subseteq \ldots$ be any ascending chain of left annihilator ideals of R . Suppose that $J_i = \ell(I_i)$ for all i . Since R is a left Quasi-Noetherian ring then there exists $m \in \mathbf{Z}^+$ such that $R^m(\bigcup_n J_n) \subseteq J_m = \ell(I_m)$, therefore $R^m(\bigcup_n J_n) I_m = ٠$, and

$R(R^{m\text{-}١}(\bigcup_n J_n)\, I_m) = ٠$. But $r(R) = ٠$, hence $R^{m\text{-}١}(\bigcup_n J_n)\, I_m = ٠$.

Continuing in this way we have $R(\bigcup_n J_n)\, I_m = ٠$, therefore $(\bigcup_n J_n)\, I_m = ٠$,

and $\bigcup_n J_n \subseteq \ell(I_m) = J_m$. Hence $J_m = J_{m+١} = \ldots$ and the chain terminates .

Now let $I_١ \subseteq I_٢ \subseteq \ldots \subseteq I_n \subseteq \ldots$ be any ascending chain of complement left ideals of R . Since R is a left Quasi-Noetherian ring then there exists $m \in \mathbf{Z}^+$ such that $R^m(\bigcup_n I_n) \subseteq I_m$. Now suppose that I_m is a complement of J_m then $(R^m(\bigcup_n I_n)) \cap J_m \subseteq I_m \cap J_m = ٠$. But $(\bigcup_n I_n) \cap J_m \subseteq \bigcup_n I_n$ and $(\bigcup_n I_n) \cap J_m \subseteq J_m$, hence $R^m((\bigcup_n I_n) \cap J_m) \subseteq R^m(\bigcup_n I_n)$ and $R^m((\bigcup_n I_n) \cap J_m) \subseteq (\bigcup_n I_n) \cap J_m \subseteq J_m$. Therefore

$R^m((\bigcup_n I_n) \cap J_m) \subseteq (R^m(\bigcup_n I_n)) \cap J_m = ٠$ and

$R(R^{m\text{-}١}((\bigcup_n I_n) \cap J_m)) = ٠$. But $r(R) = ٠$, hence $R^{m\text{-}١}((\bigcup_n I_n) \cap J_m) = ٠$.

Continuing in this way we have $(\bigcup_n I_n) \cap J_m = ٠$, and by maximalty of I_m, we have $\bigcup_n I_n = I_m$.Hence $I_m = I_{m+١} = \ldots$. Therefore R is a left Goldie ring.

□

Corollary ٢,٣,٣

Let R be a semi-prime left Quasi-Noetherian ring , then R has a ring of left Quotient which is a semi-simple left Artinian ring.

Proof :

Since R is a semi-prime, it follows that $r(R) = ٠$, and by Theorem ٢,٣,٢ R is a semi-prime left Goldie ring . Therefore R has a ring of left Quotient

which is a semi-simple left Artinian (by Goldie's Theorem) .

□

Corollary ٢,٣,٤

Let R be a semi-prime left Quasi-Noetherian ring . Then every annihilator ideal in R is a finite intersection of minimal prime ideals of R.

Proof :

Since R is a left Goldie ring , it follow from Theorem ١,١,٣٥ that every annihilator ideal in R is a finite intersection of minimal prime ideals in R.

□

Next we prove the following which gives the relationship between left Quasi-Noetherian ring and its ring of left quotient .

Theorem ٢,٣,٥

Let R be a left Quasi-Noetherian ring with at least one regular element then its ring of left quotient $Q(R)$ is a left Quasi-Noetherian ring .

Proof :

Let $J_١ \subseteq J_٢ \subseteq ... \subseteq J_n \subseteq ...$ be any ascending chain of left ideals of $Q(R)$, then $J_١ \cap R \subseteq J_٢ \cap R \subseteq ... \subseteq J_n \cap R \subseteq ...$ is an ascending chain of left

ideals of R , so there exists $m \in \mathbf{Z}^{+}$ such that $R^{m}(\bigcup_{n} J_{n} \cap R) \subseteq J_{m} \cap R$.

Since $Q(R)$ is a ring of left quotient of R then every regular element of R is invertible in $Q(R)$. So let S be the subset of R which contain every regular element of R so $S^{-١} \subseteq Q(R)$ and $S^{-١}S = ١$.

Now $S^{m}(\bigcup_{n} J_{n} \cap R) \subseteq R^{m}(\bigcup_{n} J_{n} \cap R) \subseteq J_{m} \cap R$

$$(Q(R))^{٢m} S^{m}(\bigcup_{n} J_{n} \cap R) \subseteq (Q(R))^{٢m}(J_{m} \cap R) \subseteq Q(R)(J_{m} \cap R) = J_{m}$$

$$(Q(R))^{m} S^{-m} S^{m}(\bigcup_{n} J_{n} \cap R) \subseteq (Q(R))^{٢m} S^{m}(\bigcup_{n} J_{n} \cap R) \subseteq J_{m}$$

$$(Q(R))^{m-١} Q(R)(\bigcup_{n} J_{n} \cap R) \subseteq J_{m}$$

$$(Q(R))^{m}(\bigcup_{n} J_{n}) \subseteq J_{m}$$

Hence $Q(R)$ is a left Quasi-Noetherian ring .

□

Remark ٢,٣,٦

The converse of Theorem (٢,٣,٥) is not necessarily true as the following example shows :

Let $R = F[x_{١}, x_{٢}, \ldots]$ be a polynomial ring with infinitely many indeterminates over the field of characteristic zero F . Then its ring of quotient $Q(R) = K$, which is a field and it's Noetherian and so Quasi-Noetherian but R is not .

Next we prove a small generalization of Hilbert's Basis Theorem , namely the following .

<u>Theorem ٢,٣,٧</u> :

Let R be a left Quasi-Noetherian ring without zero divisors . Then $R[x]$ is a left Quasi-Noetherian .

To prove this we need the following .

<u>Lemma ٢,٣,٨</u> :

Let R be a ring without zero divisors and J, K are left ideals in the polynomial ring $R[x]$, $J \subseteq K$ and $t \geq ٠$.

If $L_t(J)=\{a_t \in R \mid \exists a_٠, a_١, ..., a_{t-١}$ such that $a_٠ + a_١x + ... + a_tx^t \in J\}$. Then

(a) $L_t(J)$ is a left ideal in R and $L_t(J) \subseteq L_{t+١}(J)$ for all $t \geq ٠$.

(b) $L_t(J) \subseteq L_t(K)$ for all $t \geq ٠$,

(c) If there exists $m \in \mathbf{Z}^+$ such that $R^m L_t(K) \subseteq L_t(J)$ for all $t \geq ٠$, then $(R[x])^m K \subseteq J$.

<u>Proof</u> :

(a) [١١, p ٤١٨] . Since R is a subring of $R[x]$, it follows that $r \in R[x]$ for all $r \in R$. Hence for every $r \in R$ we have $r(a_٠ + a_١x + ... + a_tx^t) \in J$ (since $J \lhd_\ell R[x]$) , therefore $ra_t \in L_t(J)$ for

all $a_t \in L_t(J)$, hence $L_t(J) \lhd_\ell R$, and since $f_t = a_{٠} + a_{١}x + ... + a_t x^t \in J$ implies that $xf_t = a_{٠}x + a_{١}x^{٢} + ... + a_t x^{t+١} \in J$, then $L_t(J) \subseteq L_{t+١}(J)$.

(b) Let $a_t \in L_t(J)$ for all $t \geq ٠$. Then there exists $a_{٠}, a_{١}, ..., a_{t-١}$ such that $a_{٠} + a_{١}x + ... + a_t x^t \in J$ but $J \subseteq K$ so $a_t \in L_t(K)$ for all $t \geq ٠$,therefore $L_t(J) \subseteq L_t(K)$ for all $t \geq ٠$.

(c) Let $f(x) \in (R[x])^m K$, $\deg(f(x)) = t \geq ٠$. Then

$$L_t(f(x)) \in L_t((R[x])^m K)$$

$$= L_r((R[x])^m)\, L_{t-r}(K)$$

$$= L_{r_١}(R[x]) L_{r_٢}(R[x]) .... L_{r_m}(R[x]) L_{t-r}(K) \quad , \sum_{i=١}^{m} r_i = r$$

$$\subseteq R^m L_{t-r}(K) \subseteq L_{t-r}(J) \subseteq L_t(J) .$$

Therefore $f(x) \in J$. Hence $(R[x])^m K \subseteq J$.

□

Lemma ٢,٣,٩ :

$$\bigcup_n L_t(I_n) = L_t(\bigcup_n I_n)$$

Proof :

Let $a_t \in \bigcup_n L_t(I_n)$ .Then $a_t \in L_t(I_n)$ for some n , hence there exists $a_{٠}, ..., a_{t-١}$ such that $a_{٠} + a_{١}x + ... + a_t x^t \in I_n$ for some n . Therefore

$a_٠ + a_١ x + ... + a_t x^t \in \bigcup_n I_n$ and $a_t \in L_t(\bigcup_n I_n)$. Hence $\bigcup_n L_t(I_n) \subseteq L_t(\bigcup_n I_n)$.

Now let $b_t \in L_t(\bigcup_n I_n)$. Then there exists $b_٠,...,b_{t-١}$ such that $b_٠ + b_١ x + ... + b_t x^t \in \bigcup_n I_n$ .Therefore $b_٠ + b_١ x + ... + b_t x^t \in I_n$ for some n, and $b_t \in L_t(I_n)$ for some n , hence $b_t \in \bigcup_n L_t(I_n)$, and $L_t(\bigcup_n I_n) \subseteq \bigcup_n L_t(I_n)$. Therefore $\bigcup_n L_t(I_n) = L_t(\bigcup_n I_n)$.

□

<u>Proof of theorem ٢,٣,٧</u> :

Let $I_١ \subseteq I_٢ \subseteq ... \subseteq I_n \subseteq ...$ be any ascending chain of left ideals of $R[x]$. Then for all $t \geq ٠$, $L_t(I_١) \subseteq L_t(I_٢) \subseteq ... \subseteq L_t(I_n) \subseteq ...$ is an ascending chain of left ideals of R . But R is a left Quasi-Noetherian ring so there exists $m \in \mathbf{Z}^+$ such that $R^m(\bigcup_n L_t(I_n)) \subseteq L_t(I_m)$ so by Lemma (٢,٣,٩) $R^m L_t(\bigcup_n I_n) \subseteq L_t(I_m)$ then by Lemma (٢,٣,٨) $(R[x])^m(\bigcup_n I_n) \subseteq I_m$ hence $R[x]$ is a left Quasi-Noetherian ring . □

An immediate consequence of Theorem ٢,٣,٧ we have the following .

<u>Corollary ٢,٣,١٠</u>

If R is a left Quasi-Noetherian ring without zero divisors , then $R[x_١,...,x_n]$ is a left Quasi-Noetherian .

□

Now we raise the following question

If R is a left Quasi-Noetherian . Is $R[[x]]$ a left Quasi-Noetherian ?

We give this a partial answer , namely the following .

Theorem ٢,٣,١١

Let R be a left Quasi-Noetherian ring without zero divisors . Then the ring of formal power series $R[[x]]$ is a left Quasi-Noetherian .

To prove this we need the following .

Lemma ٢,٣,١٢

Let R be a ring without zero divisors and J, K are left ideals in the ring of formal power series $R[[x]]$, $J \subseteq K$ and $t \geq ٠$.

If $C_t(J) = \{b_t \in R | \exists b_{t+١}, b_{t+٢}, \ldots$ such that $b_t x^t + b_{t+١} x^{t+١} + \ldots \in J\}$. Then

(i) $C_t(J)$ is a left ideal in R and $C_t(J) \subseteq C_{t+١}(J)$ for all $t \geq ٠$.

(ii) $C_t(J) \subseteq C_t(K)$ for all $t \geq ٠$,

(iii) If there exists $m \in \mathbf{Z}^+$ such that $R^m C_t(K) \subseteq C_t(J)$ for all $t \geq ٠$, then $(R[[x]])^m K \subseteq J$.

Proof :

This can be proved exactly by the same way as in Lemma (٢,٣,٨) .

□

Lemma ٢,٣,١٣

$$\bigcup_n C_t(I_n) = C_t(\bigcup_n I_n)$$

Proof :

This can be proved exactly by the same way as in Lemma (٢,٣,٩) .

□

Proof of theorem ٢,٣,١١ :

Let $I_{١} \subseteq I_{٢} \subseteq ... \subseteq I_n \subseteq ...$ be any ascending chain of left ideals of $R[[x]]$. Then for all $t \geq ٠$, $C_t(I_{١}) \subseteq C_t(I_{٢}) \subseteq ... \subseteq C_t(I_n) \subseteq ...$ is an ascending chain of left ideals of R . But R is a left Quasi-Noetherian ring so there exists $m \in \mathbf{Z}^{+}$ such that $R^m(\bigcup_n C_t(I_n)) \subseteq C_t(I_m)$ so by Lemma (٢,٣,١٣) $R^m C_t(\bigcup_n I_n) \subseteq C_t(I_m)$ then by Lemma (٢,٣,١٢) $(R[[x]])^m(\bigcup_n I_n) \subseteq I_m$, hence $R[[x]]$ is a left Quasi-Noetherian ring .

□

Corollary ٢,٣,١٤

Let R be a left Quasi-Noetherian ring without zero divisors , then the ring of extended formal power series $R\langle x\rangle$ is a left Quasi-Noetherian .

Proof :

By Theorem ٢,٣,٧ $R[x]$ is a left Quasi-Noetherian ring . But $R\langle x\rangle$ is a ring of left quotient of $R[x]$ so by Theorem ٢,٣,٥ $R\langle x\rangle$ is a left Quasi-Noetherian ring .

□

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بسم الله الرحمن الرحيم

خلاصة البحث

يعتبر تحديد بنية الحلقيات والحلقات النوثيرية واحدة من أهم المشكلات في نظرية الحلقيات وقد دلت الدراسات المتتالية لتلك الحلقيات والحلقات على إمكانية دراسة حلقيات أعم. وعليه فقد درسنا في هذا البحث نوع جديد من الحلقيات والحلقات يملك العديد من الخواص الأساسية للحلقيات (الحلقات) النوثيرية أطلقنا عليه ***الحلقيات شبه النوثيرية اليسرى*** (يقال عن حلقية يسرى M على حلقة R *أنها شبه نوثيرية* (Quasi-Noetherian Module) إذا كان لكل سلسلة متصاعدة

$N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ من الحلقيات الجزئية من M يوجد عدد صحيح موجب m ($m \in Z^+$) بحيث أن ($R^m(\bigcup_n N_n) \subseteq N_m$) .

ويقال عن R أنها ***حلقة شبه نوثيرية يسرى*** إذا كان $_R R$ حلقية شبه نوثيرية.

لاحظ أن هذا النوع من الحلقات هو تعميم للحلقات النوثيرية اليسرى و الحلقات المتلاشية .

وتضم هذه الرسالة **فصلين** :

يتضمن **الفصل الأول** منها بعض التعاريف و النتائج الأساسية المعروفة في نظرية الحلقات والتي استخدمت في هذا البحث .

أما **الفصل الثاني** فقد ضم **ثلاثة بنود** تناولنا *في الأول* منها مفهوم الحلقيات والحلقات شبه النوثيرية اليسرى موضحين ذلك ببعض الأمثلة ومن ثم أوجدنا الشروط المكافئة للتعريف ، ثم درسنا العلاقة بين الحلقيات النوثيرية اليسرى و الحلقيات شبه النوثيرية اليسرى وبصورة خاصة أثبتنا في مبرهنة (٢,١,٦) أنه إذا كانت RM حلقية نوثيرية يسرى فإن M حلقية شبه نوثيرية يسرى . ثم بينا في مبرهنة (٢,١,٨) أنه إذا كان $\mathcal{M}$ فصل (class) جميع الحلقيات شبه النوثيرية اليسرى، فإن $\mathcal{M}$ مغلق بالنسبة للباقي (Q-closed) أي أنه إذا كان $M \in \mathcal{M}$ و N حلقية جزئية من M فإن

$M/N \in \mathcal{M}$. كما أن $\mathcal{M}$ مغلق بالنسبة للأجزاء (S-closed) أي أنه إذا كان $\mathcal{M} \ni M$ و N حلقية جزئية من M فإن $N \in \mathcal{M}$. ثم أثبتنا في مبرهنة (٢,١,١٢) أن $\mathcal{M}$ مغلق بالنسبة للتوسع (E-closed) أي أنه إذا كانت Mحلقية و N حلقية جزئية منها و كان $M/N, N \in \mathcal{M}$ فإن $M \in \mathcal{M}$. وإذا كان $\mathfrak{X}$ فصل جميع الحلقات شبه النوثيرية اليسرى فقد بينا أن $\mathfrak{X}$ غير مغلق بالنسبة للأجزاء (not S-closed) أي أنه إذا كانت $R \in \mathfrak{X}$ و S حلقة جزئية من R فلا يشترط أن تكون $S \in \mathfrak{X}$ ، كما أن $\mathfrak{X}$ غير مغلق بالنسبة للتوسع (not E-closed) أي أنه إذا كانت R حلقة و I مثالية فيها و كان $I \in \mathfrak{X}$ ، R/I فلا

يشترط أن تكون $R \in \mathfrak{X}$، وأثبتنا في مبرهنة(٢,١,٨) و مبرهنة (٢,١,١٥) أن فصل الحلقات شبه النوثيرية اليسرى مغلق بالنسبة للباقي(Q-closed) وبالنسبة للمثاليات(I-closed) أي أنه

(أ) إذا كانت $R \in \mathfrak{X}$ و I مثالية في R فإن $R/I \in \mathfrak{X}$.

(ب) إذا كانت $R \in \mathfrak{X}$ ، $I \lhd_{\ell} R$ فإن $I \in \mathfrak{X}$.

كما أثبتنا في مبرهنة (٢,١,١٧) أن الجمع المباشر لعدد منتهي من الحلقات شبه النوثيرية اليسرى هو حلقة شبه نوثيرية يسرى، وأخيرا أثبتنا في مبرهنة (٢,١,١٨) أنه إذا كانت R حلقة شبه نوثيرية يسرى فإنها تحقق خاصية السلسة المتصاعدة (a.c.c) على المثاليات الشبة الأولية .

أما *البند الثاني* فقد خصص لدراسة بنية المثاليات في الحلقات شبه النوثيرية اليسرى حيث أثبتنا في مبرهنة(٢,٢,١) أن كل مثالية يسرى شبه متلاشية في حلقة شبه نوثيرية يسرى تكون مثالية متلاشية . ثم أثبتنا في مبرهنة (٢,٢,٤) أن الحلقة R تكون حلقة شبه نوثيرية يسرى إذا وإذا فقط كان $N=N(R)$ (الجذر الشبه متلاشي للحلقة R) مثالية متلاشية وكل من $R/N,\ N/N^{2},\ N^{2}/N^{3}, \ldots$ حلقيات شبه نوثيرية يسرى على الحلقة R . كما أثبتنا في مبرهنة (٢,٢,٥)أن كل مثالية غير قابله للتحليل في الحلقة الإبدالية شبة النوثيرية تكون مثالية ابتدائية . ثم أثبتنا في مبرهنة (٢,٢,٦) أن كل مثالية في حلقة إبدالية شبه نوثيرية محتواه في تقاطع عدد

منتهي من المثاليات الابتدائية. ثم أثبتنا في مبرهنة (٢,٢,٨) أنه إذا كانت R حلقة غير متلاشية شبه نوثيرية يسرى فإنها تملك مثالية أولية تالفة

(a prime ideal which is a right annihilator of some non-zero ideal in R)

و أخيرا أثبتنا في مبرهنة (٢,٢,٩) أن كل مثالية في الحلقة شبه النوثيرية اليسرى تحوي حاصل ضرب مثاليات أولية .

أما في *البند الثالث* من هذا الفصل فقد تطرقنا لدراسة علاقة الحلقات شبه النوثيرية اليسرى بأنواع خاصة من الحلقات كحلقات جولدي (Goldie Rings) و حلقة البواقي (Ring of Quotient) و حلقة كثيرات الحدود (Polynomial Ring) حيث أثبتنا في مبرهنة (٢,٣,٢) أنه إذا كانت R حلقة شبه نوثيرية يسرى وكان التالف الأيمن للحلقة R يساوي الصفر ($r(R)=٠$) فإن R حلقة جولدية يسرى . ثم برهنا في مبرهنة (٢,٣,٥) أنه إذا كانت R حلقة شبه نوثيرية يسرى تملك عنصر منتظم (regular element) واحد على الأقل فإن حلقة البواقي لها $Q(R)$ تكون حلقة شبه نوثيرية يسرى. ثم أثبتنا في مبرهنة (٢,٣,٧) و مبرهنة(٢,٣,١١) أنه إذا كانت R حلقة شبه نوثيرية يسرى خالية من القواسم الصفرية فإن حلقة كثيرات الحدود لها$R[x]$ وكذلك حلقة متسلسلات القوى لها $R[[x]]$ حلقات شبه نوثيرية يسرى .

وأخيرا نود أن نشير إلى أنه قد أرسل بحثين للنشر من هذه الرسالة والله الموفق لما يحبه ويرضاه ، وأخر دعوانا أن الحمد لله رب العالمين .

شكر وتقدير

الحمد لله رب العالمين ملهم كل صواب وولي كل خير، خلق الإنسان وشرفه بحمل أمانة العقل وعلمه البيان وقال أعز من قائل ﴿ **علم الإنسان ما لم يعلم** ﴾. فأحمده تعالى على توفيقه وأشكره على نعمائه وأصلي وأسلم على الهادي الأمين سيد المرسلين وخير الخلائق أجمعين محمد بن عبد الله وعلى آله وصحبه الطاهرين.

وبعد: امتثالا لقول رسولنا الكريم ﷺ (من لا يشكر الناس لا يشكر الله)

فإنني أتقدم بخالص شكري وفائق تقديري و جميل عرفاني لسعادة الأستاذ الدكتور :

فالح بن عمران بن محمد الدوسري الذي تفضل مشكورا باقتراح موضوع الرسالة ، ولم يبخل علي بعلمه ووقته وجهده المثمر وأعلم أنه مهما سطرت من كلمات شكر فلن أفيه حقه ولكن هو جهد المقل فله مني موفور الشكر على اقتراح الموضوع ومتابعته وتوجيهه بملاحظاته القيمة شكرا موصولا بدعوات حارة بأن يجزيه الله خير الجزاء وأن يبارك له في صحته ووقته وعلمه وينفع به الأنام .

ولا يفوتني أن أقدم جزيل شكري لجامعة أم القرى لاحتضانها لي كطالبة ماجستير، كذلك أتوجه بالشكر الجزيل لقسم العلوم الرياضية بجميع أعضاءه على تعاونهم وتهيئة الظروف للبحث والدراسة ، كما أبعث رسالة شكر إلى الأساتذة الكرام الذين قاموا بتدريسي في مرحلة الماجستير فقد كان لهم فضل كبير في مسيرتي الدراسية والعلمية وأدعو الله العلي القدير أن يجزيهم خير الجزاء على ما قدموا من علم وعمل .

كما أود أن أعرب عن شكري وامتناني لأسرتي الكريمة فالتشجيع والدعم الذي أبدوه لي هو ما أثمنه أعظم تثمين، وأخص بالشكر والدي الكريمين ومن أولتني الرعاية والاهتمام فلولا دعواتهم لم أكن لأصل لهذا المستوى العلمي وهنا تقف حروفي عاجزة

عن تنظيم عقد من جمل الشكر لهم وكل ما أستطيعه هو الدعاء لهم بأن يجزيهم الله خير الجزاء وأن ينعم عليهم بموفور الصحة والعافية .

والشكر موصول لكل من بادلني مشاعره الطيبة وشجعني وساعدني في مسيرتي العلمية من أصدقاء وزملاء وفي هذا المجال أقدم باقة من الشكر والامتنان لرفيقات هذه المرحلة الأخوات **:** ***أميمه الشنقيطي*** و ***إيمان اللقماني*** فبارك الله فيهم وسدد على الخير خطاهم وبلغهم آمالهم .

وأخيرا أسأل المولى عز وجل أن يتقبل هذا العمل والجهد واسأله أن ينفع به ويجعله خالصا لوجهه الكريم فهو حسبنا ونعم الوكيل وصلى الله وسلم على سيدنا و نبينا محمد وعلى أله وصحبه أجمعين وأخر دعوانا أن الحمد لله رب العالمين .

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الحلقات والحلقيات شبه النوثيرية اليسرى

الرسالة المقدمة من

أماني محمد عبد الرحيم الفضلي

كجزء متمم للحصول على درجة

الماجستير في الرياضيات

(الجبر)

إشراف

أ.د. فالح بن عمران بن محمد الدوسري

١٤٢٧هـ - ٢٠٠٦م